

EFFICIENT GEODESICS AND AN EFFECTIVE ALGORITHM FOR DISTANCE IN THE COMPLEX OF CURVES

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ABSTRACT. We give an algorithm for determining the distance between two vertices of the complex of curves. While there already exist such algorithms, for example by Leasure, Shackleton, and Webb, our approach is new, simple, and more effective for all distances accessible by computer. Our method gives a new preferred finite set of geodesics between any two vertices of the complex, called efficient geodesics, which are different from the tight geodesics introduced by Masur and Minsky.

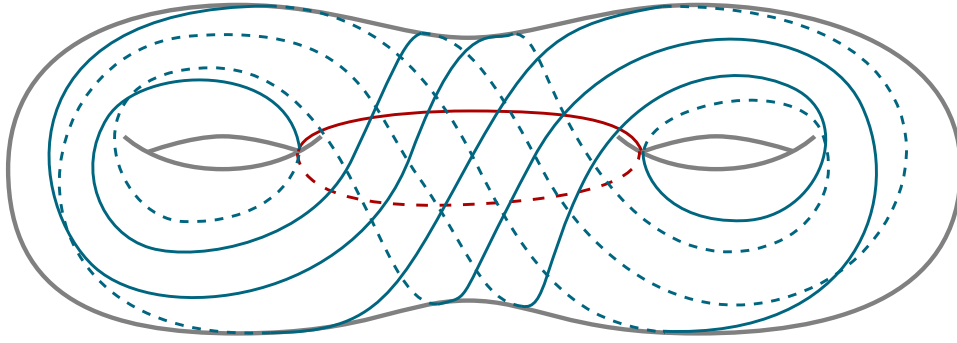


FIGURE 1. Vertices of $\mathcal{C}(S_2)$ with distance 4 and intersection number 12; this is the smallest possible intersection for vertices with distance 4

1. INTRODUCTION

The complex of curves $\mathcal{C}(S)$ for a compact surface S is the simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves in S and whose edges connect vertices with disjoint representatives. We can endow the 0-skeleton of $\mathcal{C}(S)$ with a metric by defining the distance between two vertices to be the minimal number of edges in any edge path between the two vertices.

The geometry of $\mathcal{C}(S)$ —especially the large-scale geometry—has been a topic of intense study over the past two decades, as there are deep applications to the theories of 3-manifolds, mapping class groups, and Teichmüller space; see, e.g., [14]. The seminal result, due to Masur and Minsky in 1996, states that $\mathcal{C}(S)$ is δ -hyperbolic [13]. Recently, several simple proofs of this fact have been found, and it has been shown that δ can be chosen independently of S ; see [2, 6, 7, 10, 15].

In 2002, Leasure [11, §3.2] found an algorithm to compute the distance between two vertices of $\mathcal{C}(S)$, and since then other algorithms have been devised by Shackleton [17], Webb [19], and Watanabe [18]. About his algorithm, Leasure says:

We do not mention this in the belief that anyone will ever implement it. The novelty is that finding the exact distance between two curves in the curve complex should be so awkward.

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One goal of this paper is to give an algorithm for distance—the efficient geodesic algorithm—that actually can be implemented, at least for small distances. The third author and Glenn, Morrell, and Morse [9] have in fact already developed an implementation of our algorithm, called Metric in the Curve Complex [8]. Their program is assembling a data bank of examples as we write.

Known examples. Let S_g denote a closed, connected, orientable surface of genus g and let $i_{\min}(g, d)$ denote the minimal intersection number for vertices of $\mathcal{C}(S_g)$ with distance d . The Metric in the Curve Complex program has been used to show that:

- (1) $i_{\min}(2, 4) = 12$ and
- (2) $i_{\min}(3, 4) \leq 21$.

The highly symmetric example in Figure 1—which realizes $i_{\min}(2, 4)$ —was discovered using the program. See Section 2 for a discussion of this example and a proof using the methods of this paper that the distance is actually 4.

We are only aware of one other explicit picture in the literature of a pair of vertices of $\mathcal{C}(S_2)$ that have distance four, namely, the example of Hempel that appears in the notes of Saul Schleimer [16, Figure 2] (see [9, Example 1.6] for a proof that the distance is 4). This example has geometric intersection number 25.

Using the bounded geodesic image theorem [12, Theorem 3.1] of Masur and Minsky (as quantified by Webb [21]) it is possible to explicitly construct examples of vertices with any given distance; see [17, Section 6]. We do not know how to keep the intersection numbers close to the minimum with this method, but Aougab and Taylor did in fact use this method to give examples of vertices of arbitrary distance whose intersection numbers are close to the minimum in an asymptotic sense; see their paper [3] for the precise statement.

Local infinitude. One reason why computations with the complex of curves are so difficult is that it is locally infinite and moreover there are infinitely many geodesics (i.e. shortest paths) between most pairs of vertices. Masur and Minsky [12] addressed this issue by finding a preferred set of geodesics, called tight geodesics, and proving that between any two vertices there are finitely many tight geodesics; see Section 2.2 for the definition. Our first goal is to give a new class of geodesics that still has finitely many elements connecting any two vertices but is more amenable to certain computations.

Efficient geodesics. Our approach to geodesics in $\mathcal{C}(S)$ is defined in terms of intersections with arcs. First, suppose that γ is an arc in S and α is a simple closed curve in S . We say that γ and α are in *minimal position* if α is disjoint from the endpoints of γ and the number of points of intersection of α with γ is smallest over all simple closed curves that are homotopic to α through homotopies that do not pass through the endpoints of γ .

Let v_0, \dots, v_n be a geodesic of length at least three in $\mathcal{C}(S)$, and let α_0, α_1 , and α_n be representatives of v_0, v_1 , and v_n that are pairwise in minimal position (this configuration is unique up to isotopy of S). A *reference arc* for the triple $\alpha_0, \alpha_1, \alpha_n$

is an arc γ that is in minimal position with α_1 and whose interior is disjoint from $\alpha_0 \cup \alpha_n$; such arcs were considered by Leasure [11, Definition 3.2.1].

We say that the oriented geodesic v_0, \dots, v_n is *initially efficient* if

$$|\alpha_1 \cap \gamma| \leq n - 1$$

for all choices of reference arcs γ (this is independent of the choices of α_0 , α_1 , and α_n by the uniqueness statement above). Finally, we say that $v = v_0, \dots, v_n = w$ is *efficient* if the oriented geodesic v_k, \dots, v_n is initially efficient for each $0 \leq k \leq n - 3$ and the oriented geodesic $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ is also initially efficient.

We emphasize that to test the initial efficiency of v_k, \dots, v_n we should look at reference arcs for the triple v_k, v_{k+1} , and v_n and we allow $n - k - 1$ points of intersection of (a representative of) v_{k+1} with any such reference arc.

Existence of efficient geodesics. Our main result is that efficient geodesics always exist, and that there are finitely many between any two vertices.

Theorem 1.1. *Let $g \geq 2$. If v and w are vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$, then there exists an efficient geodesic from v to w . What is more, there is an explicitly computable list of at most*

$$n^{6g-6}$$

vertices v_1 that can appear as the first vertex on an initially efficient geodesic

$$v = v_0, v_1, \dots, v_n = w.$$

In particular, there are finitely many efficient geodesics from v to w .

We emphasize that our theorem is only for closed surfaces; see the discussion on page 18 about surfaces with boundary for an explanation. We also mention that this theorem is stronger than Theorem 1.1 in the first version of this paper [4]; see Proposition 3.7 and the accompanying discussion.

Finitely many reference arcs. While *a priori* there are infinitely many reference arcs that need to be checked in the definition of initial efficiency there are in fact finitely many. Indeed, let α_0 , α_1 , and α_n be representatives of v , v_1 , and w that have minimal intersection pairwise. Since $d(v, w) \geq 3$ it follows that α_0 and α_n fill S , which means that they together decompose S into a collection of polygons. We can endow each such polygon with a Euclidean metric and replace each segment of α_1 in each polygon with a straight line segment.

There are finitely many non-rectangular polygons in the decomposition since each $2k$ -gon contributes $-(k - 2)/2$ to $\chi(S)$. And each reference arc in a rectangular region is parallel to one in a non-rectangular region. Thus in order to check initial efficiency, it is enough to consider reference arcs that lie in a non-rectangular polygonal region. Furthermore, it is enough to consider reference arcs that are straight line segments connecting the midpoints of the α_0 -edges of a polygon. Indeed, such an arc is necessarily in minimal position with α_1 and any other reference arc can be extended to such a reference arc.

In the special case that the reference arc connects the midpoints of α_0 -edges that are consecutive in a polygon, the reference arc is parallel to the α_n -edge in

between. In this case points of $\alpha_1 \cap \gamma$ are in bijection with points of $\alpha_1 \cap \alpha_n$, and so the definition of initial efficiency can be translated into a statement about intersections of α_1 with α_n ; see Proposition 3.7 below.

Finitude of efficient geodesics. The main point of Theorem 1.1 is the existence statement; the finiteness statement can be dispensed with immediately. Indeed, for any geodesic v_0, \dots, v_n let α_0, α_1 , and α_n be representatives of v_0, v_1 , and v_n that have minimal intersection pairwise. As above, α_0 and α_n decompose S_g into a collection of polygons.

If we cut S_g along α_0 we obtain a surface S'_g with two boundary components on which α_n becomes a collection of arcs. The α_n -arcs cut S'_g into a collection of even-sided polygons. We can choose reference arcs in S'_g that are disjoint from each other, that have interiors disjoint from the α_n -arcs, and that cut S'_g into hexagons. Such a collection is obtained by taking one reference arc parallel to each parallel family of arcs of α_n and then taking additional reference arcs cutting across any remaining polygons with more than six sides.

An Euler characteristic count shows that any such collection of reference arcs has $6g - 6$ elements. Also, since α_1 is disjoint from α_0 the curve α_1 is determined up to homotopy by the number of intersections it has with each reference arc. By the definition of initial efficiency, each of these intersection numbers is between 0 and $n - 1$. This gives the bound stated in Theorem 1.1.

Discussion of the proof. Our method for proving Theorem 1.1 is detailed in Section 3. Briefly, the idea is to show that if some geodesic $v = v_0, \dots, v_n = w$ is not initially efficient then we can modify v_1, \dots, v_{n-1} by surgery in order to reduce the intersection of v_1 with v_0 and v_n . The basic surgeries we use in our proof are not new. The crucial point—and our new idea—is that it is usually not possible to reduce intersection by modifying a single vertex; rather, it is often the case that we can reduce intersection by modifying a sequence of vertices all at the same time.

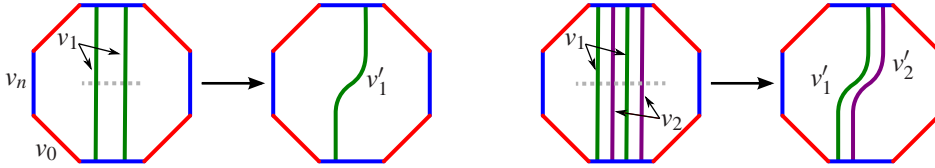


FIGURE 2. *Left:* two arcs of v_1 and a simplifying surgery; *Right:* arcs of v_1 and v_2 and a simplifying surgery

Here is what we mean by this. Suppose we have a geodesic v_0, \dots, v_n . Say there is a v_0 - v_n polygon with two parallel arcs of v_1 as in the first picture in the left-hand side of Figure 2. Then we can perform a surgery along the dotted reference arc as in the figure in order to find a vertex v'_1 that is simpler in that it has fewer intersections with v_0 and v_n . The vertex v'_1 can replace v_1 in the geodesic since the surgery did not create any intersections with v_0 or v_2 .

Now suppose we have four parallel arcs of v_1, v_2, v_1 , and v_2 (in order) as in the right-hand side of Figure 2. We cannot surger v_1 as in the previous paragraph

because this would create an intersection with v_2 —an arc of v_2 is in the way. However, we can perform surgery simultaneously on v_1 and v_2 along the dotted arc as in the figure. This gives two new vertices v'_1 and v'_2 and again we can replace v_1 and v_2 with these new, simpler vertices.

Our basic strategy is to show that whenever we have an inefficient geodesic we can find a similar surgery in order to reduce intersection with v_0 and v_n . If the reference arc only sees v_1 and v_2 then the surgeries in the previous two paragraphs apply. The problem is that when there are more vertices v_i involved, there are more and more complicated surgeries needed, and the combinatorics get to be unwieldy; look ahead to Figures 11 and 13 for examples of more complicated surgeries.

To deal with this problem, we introduce a new tool, the dot graph. This is a graphical representation of the sequence of vertices v_i seen along a reference arc; there is a dot at the point (k, i) in the plane if the k th vertex along the arc is v_i (see Figure 8 below). The existence of a simplifying surgery is translated into the existence of certain two-dimensional shapes in the dot graph (see Figure 9 below). In this way, the unwieldy combinatorial problem becomes a manageable geometric one.

Efficiency versus tightness. We already mentioned that there are finitely many tight geodesics between two vertices of $\mathcal{C}(S_g)$ and so Theorem 1.1 gives a second finite class of geodesics connecting two vertices of $\mathcal{C}(S_g)$. The next proposition shows that the class of efficient geodesics is genuinely new.

Proposition 1.2. *Let $g \geq 2$. In $\mathcal{C}(S_g)$ there are geodesics of length three that are...*

- (1) *efficient and tight,*
- (2) *tight but not efficient, and*
- (3) *efficient but not tight.*

We do not know if between any two vertices there always exists a geodesic that is efficient and tight.

Proposition 1.2 is proved by explicit construction; see Section 2.2. The most subtle point is the third one, as it is in general not easy to prove that a given geodesic is not contained in any tight multigeodesic.

While the examples of geodesics in Proposition 1.2 all have length three, we expect that the result holds for all distances at least three. It is also worth noting that our constructions are all delicate: it is not obvious how to modify our examples in order to obtain infinite families of examples.

The efficient geodesic algorithm. We now explain how Theorem 1.1 can be used in order to give an algorithm for distance in $\mathcal{C}(S_g)$, which we call the *efficient geodesic algorithm*. It is straightforward to determine if the distance between two vertices is 0, 1, or 2. So assume that for some $k \geq 2$ we have an algorithm for determining if two vertices of $\mathcal{C}(S_g)$ have distance $0, \dots, k$. We would like to give an algorithm for determining if the distance between two vertices is $k + 1$.

To this end, let v and w be two vertices of $\mathcal{C}(S_g)$. By induction we can check if $d(v, w) \leq k$. If not, then as in Theorem 1.1 we can explicitly list all possible vertices v_1 on an efficient geodesic $v = v_0, \dots, v_{k+1} = w$. If $d(v_1, w) = k$ for some choice of

v_1 , then $d(v, w) = k + 1$; otherwise it follows from Theorem 1.1 (the existence of efficient geodesics) that $d(v, w) \neq k + 1$.

Corollary 1.3. *The efficient geodesic algorithm computes distance in $\mathcal{C}(S_g)$.*

The special case of the efficient geodesic algorithm when the distance is four was explained to us by John Hempel and served as inspiration for the cases of larger distance.

Comparison with previously known algorithms. Our efficient geodesic algorithm is in the same spirit as the algorithms of Leasure, Shackleton, and Watanabe for computing distance in $\mathcal{C}(S_g)$. All three show that there is a function F of three variables so that for any two vertices v and w of $\mathcal{C}(S_g)$ with $d(v, w) = n$ there is a geodesic $v = v_0, \dots, v_n = w$ with $i(v_1, w)$ bounded above by $F(g, n, i(v, w))$. This gives an algorithm in the same way as our efficient geodesic algorithm, since there is an explicitly computable list of v_1 with $i(v, v_1) = 0$ and $i(v_1, w) \leq F(g, n, i(v, w))$. While the theorems of Leasure, Shackleton, and Watanabe apply to surfaces that are not closed, we restrict here to the case of closed surfaces for simplicity.

Our approach also gives such a function F . By only considering reference arcs that are parallel to arcs of $\alpha_n \setminus \alpha_0$ (where α_0 and α_n are minimally-intersecting representatives of v_0 and v_n), we deduce that for any initially efficient geodesic $v = v_0, \dots, v_n = w$ we have $i(v_1, v_n) \leq (n - 2)i(v, w)$ (this uses a slight strengthening of a special case of Theorem 1.1; see Proposition 3.7 below). So we can take

$$F_{BMM}(g, n, i(v, w)) = (n - 2)i(v, w).$$

However, this bound does not use the full strength of initial efficiency as it does not give information as to how these points of intersection are distributed along α_n nor does it take into account reference arcs that are not parallel to α_n .

Leasure's function is

$$F_L(g, n, i(v, w)) = (6(6g - 2) + 2)^n i(v, w).$$

We can illustrate the improvement of our methods over Leasure's with the example in $\mathcal{C}(S_2)$ from Figure 1. To prove the distance is 4, we can suppose for contradiction that it is 3. According to Leasure, if v_1 is the first vertex we meet on a length 3 geodesic from v to w , then we can choose v_1 so that it satisfies

$$i(v_1, w) \leq (6(6g - 2) + 2)^3 i(v, w) = 62^3 \cdot 12 = 2,859,936.$$

By contrast, any v_1 on an efficient geodesic of length 3 satisfies $i(v_1, w) \leq 12$ and, what is more, we know there is at most one intersection of v_1 along each edge of the polygonal decomposition of S_2 determined by v and w (cf. Proposition 3.7 below). Because of these strong restrictions, the computation can be carried out by hand, and in fact we apply the algorithm by hand to this example in Section 2.

Shackleton's function depends only on $i(v, w)$ and g , but not $d(v, w)$. As explained by Watanabe [18], Shackleton's function is

$$F_S(g, n, i(v, w)) = i(v, w)(4^5(3g - 3)^3)^{2 \log_2 i(v, w)}.$$

Watanabe recently improved on Shackleton's result by replacing the exponential function with a linear one. His work, like Webb's, uses the theory of tight geodesics. Specifically, Watanabe's function is:

$$F_W(g, n, i(v, w)) = R_g i(v, w)$$

where $R_g = (3g - 3) \cdot 2^{(3M+1)^3(2g-2)^{3g-3}}$ and M is the minimal possible constant in the bounded geodesic image theorem. Since R_g is independent of n , it follows that when n is large compared to g Watanabe's bounds give a better algorithm for distance than the efficient geodesic algorithm. However, the smallest known upper bound for M is 102 (see [22]), and so even for $g = 2$, we have

$$R_g = 3 \cdot 2^{231,475,544} > 10^{69,681,082}.$$

Thus, even for $g = 2$ and some unimaginably large distances, our algorithm is more effective.

In the appendix we will explain Webb's algorithm for computing distance via tight geodesics. As explained to us by Webb [20], his methods give a corresponding function that again only depends on g :

$$F_{W'}(g) = \frac{(6g - 6) ((4g - 5)^{21} - 4g + 5)}{2g - 3},$$

which for $g = 2$ equals 62,762,119,200.

A more appropriate comparison with Webb's algorithm is to compare the number of vertices v_1 that need to be tested instead of the quantity $i(v_1, v_n)$. In Webb's algorithm, this number is bounded above by:

$$2^{(72g+12)\min\{n-2, 21\}} (2^{6g-6} - 1)$$

(here we are really counting the number of candidate simplices σ_1 along a multi-geodesic from v to w); see the appendix of this paper for an explanation. On the other hand, our Theorem 1.1 states that the number of candidate vertices v_1 along an efficient geodesic v_0, \dots, v_n is bounded above by n^{6g-6} . Our bound is smaller than Webb's when $\min\{n - 2, 21\} = n - 2$. In the case that $\min\{n - 2, 21\} = 21$ we estimate Webb's bound from below by $2^{(72g)(21)}$ and we find that our bound is smaller than Webb's for all distances less than $2^{21(12)}$, which is approximately 10^{75} . We conclude that among all known algorithms for distance in $\mathcal{C}(S)$ our methods are by far the most effective for all distances accessible by modern computers.

Acknowledgments. We would like to thank Ken Bromberg, Chris Leininger, Yair Minsky, Kasra Rafi, and Yoshuke Watanabe for helpful conversations. We are especially grateful to John Hempel for sharing with us his algorithm, to Richard Webb for sharing many ideas and details of his work, and to Tarik Aougab for many insightful comments, especially on the problem of constructing geodesics that are not tight. Finally, we would like to thank Paul Glenn, Kayla Morrell, and Matthew Morse for supplying numerous examples generated by their program Metric in the Curve Complex.

2. EXAMPLES

In this section we do two things. First we illustrate the efficient geodesic algorithm by applying it to the example from Figure 1. Then we prove Proposition 1.2 by giving explicit examples for each of the three statements. All of the examples will be presented in terms of the branched double cover of S_g over the sphere, which we now explain.

The branched double cover. Let X_{2g+2} denote a sphere with $2g+2$ marked points. The double cover branched over the marked points is the closed surface S_g . The preimage of a simple arc in X_{2g+2} connecting two marked points is a nonseparating simple closed curve in S_g , and the preimage of a simple closed curve that surrounds $2k+1$ marked points is a separating simple closed curve in S_g that cuts off a subsurface of genus k .

Minimally intersecting curves and arcs in X_{2g+2} lift to minimally intersecting curves and arcs in S_g . This follows from the work of the first author and Hilden on the symmetric mapping class group [5]; see also the paper by Winarski [23]. Also, if two minimally intersecting curves or arcs fill X_{2g+2} —meaning that the complementary components are all disks with at most one marked point each—then the preimages fill S_g since the preimage of a disk with at most one marked point is a disk.

2.1. An example of the efficient geodesic algorithm. Consider the two arcs δ and ε in X_6 shown in the left-hand side of Figure 3 (we depict X_{2g+2} by drawing $2g+2$ dots in the plane; by adding an unmarked point at infinity, we obtain the sphere with $2g+2$ marked points). Let v and w denote the corresponding vertices of S_2 , the two-fold branched cover over X_6 . We would like to show that the distance between v and w in $\mathcal{C}(S_2)$ is 4 (it so happens that v and w are the same as the vertices of $\mathcal{C}(S_2)$ shown in Figure 1, but we will not need this). The distance between v and w in $\mathcal{C}(S_2)$ can be computed with the computer program Metric in the Curve Complex, but here we explain how to apply our algorithm by hand.

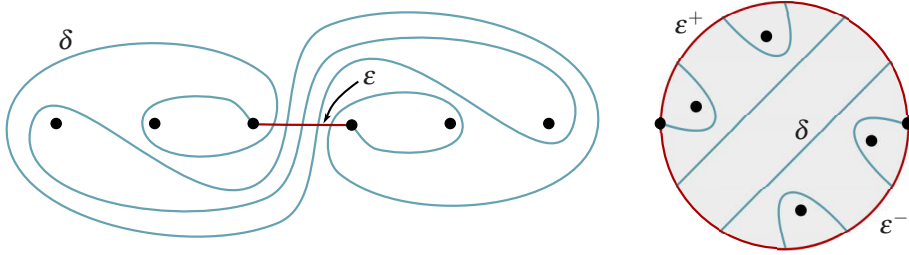


FIGURE 3. *Left:* the arcs δ and ε in X_6 corresponding to the curves shown in Figure 1; *Right:* the disk Δ obtained by cutting along ε

First, we will show that $d(v, w) \leq 4$. To do this, we observe that the horizontal line segment connecting the second and third marked points in the left-hand side of Figure 3 corresponds to a vertex u in $\mathcal{C}(S_2)$ with $i(u, v) = i(u, w) = 1$. It follows that $d(u, v) = d(u, w) = 2$ and by the triangle inequality that $d(v, w) \leq 4$.

If we cut X_6 along ε , we obtain a disk Δ , the shaded disk in the right-hand side of Figure 3. The boundary of Δ consists of two copies of ε , say, ε^+ and ε^- , and in the figure points of ε^+ and ε^- are identified in X_6 exactly when they lie on the same vertical line. The arc δ becomes a collection of arcs in Δ as shown in the figure. Since the arcs of δ cut Δ into a disjoint union of disks with at most one marked point each, it follows that δ and ε fill S_2 and so $d(v, w) \geq 3$.

It remains to use the efficient geodesic algorithm to show that $d(v, w) \geq 4$. Assume that $d(v, w)$ were equal to three. By Theorem 1.1 there is a path v, v_1, v_2, w so that the number of intersections of v_1 with each arc of $w \setminus v$ is at most two (consider a reference arc parallel to the arc of $w \setminus v$). Proposition 3.7 below gives an improvement: there is a choice of v_1 so that the intersection with each arc of $w \setminus v$ is at most one point. Also, since v, v_1, v_2, w is a path, this choice of v_1 satisfies $d(v_1, w) \leq 2$; in other words, (representatives of) v_1 and w do not fill S_2 .

A special feature of the genus two case is that every vertex of $\mathcal{C}(S_2)$ is obtained as the preimage of a curve or arc in X_6 (this again follows from the work of the first author with Hilden). In this way, any v_1 as in the previous paragraph corresponds to an arc or curve β in Δ that intersects each arc of δ in at most one point. There are only six such candidates for β , namely the six straight line segments connecting marked points in the interior of Δ . It is straightforward to check that the arc in X_6 corresponding to each fills with δ . Therefore there is no v_1 as in the previous paragraph and we have $d(v, w) = 4$.

2.2. Efficiency versus tightness. We will now prove Proposition 1.2—that there are geodesics in $\mathcal{C}(S_g)$ that are efficient and tight, geodesics that are efficient but not tight, and geodesics that are tight but not efficient. First we recall the definition of a tight geodesic.

Tight geodesics. A *tight multigeodesic* is a sequence of simplices $\sigma_0, \dots, \sigma_n$ in $\mathcal{C}(S)$ where

- (1) σ_0 and σ_n are vertices,
- (2) the distance between v_i and v_j is $|j - i|$ whenever $i \neq j$ and v_i and v_j are vertices of σ_i and σ_j , respectively, and
- (3) for each $1 \leq i \leq n - 1$ the simplex σ_i can be represented as the union of the essential components of the boundary of a regular neighborhood in S of minimally-intersecting representatives of σ_{i-1} and σ_{i+1} .

This definition is due to Masur and Minsky.¹ We will refer to any sequence of vertices v_0, \dots, v_n with $v_i \in \sigma_i$ as a *tight geodesic*.

Proof of Proposition 1.2. We begin with the first statement, there there are geodesics in $\mathcal{C}(S_g)$ that are both efficient and tight. Consider the arcs $\delta_0, \delta_1, \delta_2$, and δ_3 in X_{12} shown in Figure 4. As above, each arc δ_i represents a vertex v_i of $\mathcal{C}(S_5)$. We have $d(v_0, v_3) = 3$ since δ_0 and δ_3 fill X_{12} .

To see that the geodesic v_0, v_1, v_2, v_3 is efficient we first note that δ_1 intersects only two regions of X_{16} determined by δ_0 and δ_3 . One of these regions is a bigon

¹Masur and Minsky used the term “tight geodesic,” instead of “tight multigeodesic,” language we prefer to avoid because the object in question is not a geodesic.

with one marked point; the preimage of this is a rectangular region in S_5 and so it can be ignored. The other region is a disk with no marked points and its preimage is a pair of disks in S_5 . The preimage of δ_1 passes through each of these disks in S_5 once, whence the initial efficiency of v_0, v_1, v_2, v_3 . There is an obvious symmetry of X_{12} reversing the geodesic and so v_3, v_2, v_1, v_0 is initially efficient. Hence v_0, v_1, v_2, v_3 is indeed efficient.

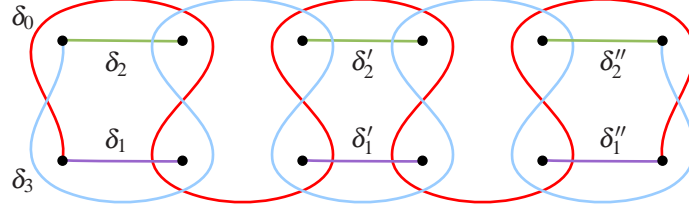


FIGURE 4. Arcs giving a geodesic in $\mathcal{C}(S_5)$ that is both efficient and tight

Let v'_i and v''_i denote the vertices of $\mathcal{C}(S_5)$ corresponding to the arcs δ'_i and δ''_i . The simplices $\sigma_1 = \{v_1, v'_1, v''_1\}$ and $\sigma_2 = \{v_2, v'_2, v''_2\}$ give a tight multigeodesic $v_0, \sigma_1, \sigma_2, v_3$ with $v_1 \in \sigma_1$ and $v_2 \in \sigma_2$, certifying that v_0, v_1, v_2, v_3 is a tight geodesic. (To verify this, note that the preimage in S_g of a disk with two marked points in X_{2g+2} is an annulus.)

For any odd $g > 2$ a straightforward generalization applies. For even g a slight modification is needed; for instance to obtain an analogous example for S_4 from Figure 4, we move the left-hand endpoints of δ_0 and δ_3 together and we move the right-hand endpoints together as well, giving a collection of arcs in X_{10} .

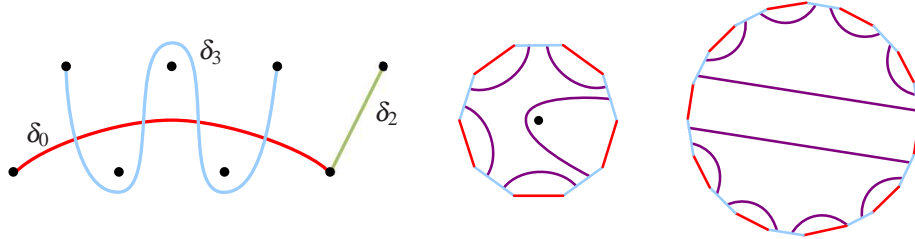


FIGURE 5. *Left:* Arcs in X_8 giving a tight geodesic in $\mathcal{C}(S_3)$; *Middle:* The outer 10-gon in X_8 cut along δ_0 and δ_3 shown with arcs of δ_1 ; *Right:* The preimage of the 10-gon in S_3 shown with the preimage of δ_1

We now give examples of geodesics that are tight but not efficient. Consider the arcs δ_0 , δ_2 , and δ_3 in shown in the left-hand side of Figure 5. Let δ_1 be the boundary of a regular neighborhood of $\delta_0 \cup \delta_2$; this δ_1 is a curve surrounding three marked points. Let v_0, v_1, v_2, v_3 be the corresponding path in $\mathcal{C}(S_3)$. We have $d(v_0, v_3) \geq 3$ since δ_0 and δ_3 fill X_8 . By definition v_0, v_1, v_2, v_3 is tight at v_1 (meaning that the third part of the definition of a tight multigeodesic is satisfied for $i = 1$) and it is straightforward to check that it is tight at v_2 ; so more than being contained in a tight

multigeodesic, the given geodesic is itself a tight multigeodesic (in other words, v_0, v_1, v_2, v_3 is a tight multigeodesic with a single associated tight geodesic).

We will now show that the oriented geodesic v_0, v_1, v_2, v_3 is not initially efficient. If we cut X_8 along δ_0 and δ_3 there is a single region that is not a bigon with one marked point, namely, the region containing the (unmarked!) point at infinity. There are five arcs of δ_1 in this disk as shown in the middle picture of Figure 5 (the exact configuration relative to the marked point is important here). The preimage of this 10-gon in S_3 is a 20-gon, and the arcs of the preimage of δ_1 are arranged as in the right-hand side of Figure 5. It is easy to find a reference arc in this polygon that intersects the preimage of δ_1 in more than two points. Thus v_0, v_1, v_2, v_3 is not initially efficient; of course this implies that v_0, v_1, v_2, v_3 is not efficient. The generalization to higher genus should be clear.

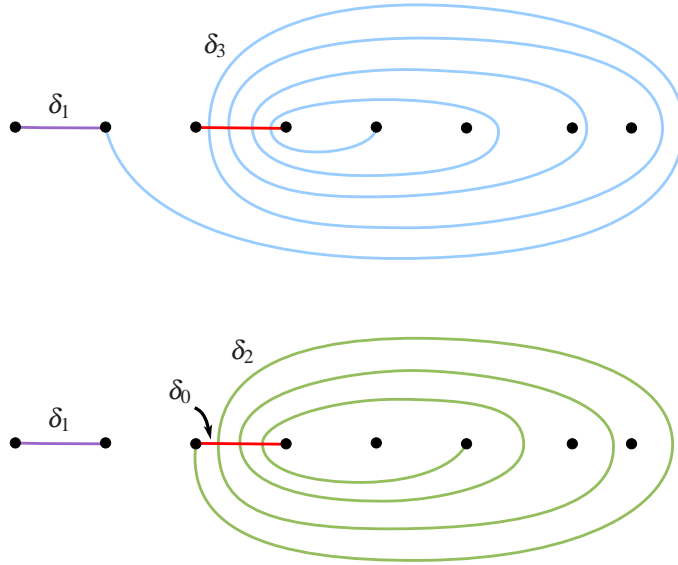


FIGURE 6. Arcs giving an efficient geodesic in $\mathcal{C}(S_3)$ that is not tight

Finally we give examples of geodesics that are efficient but not tight. Consider the arcs δ_0 , δ_1 , δ_2 , and δ_3 shown in Figure 6 (the arcs δ_0 , δ_1 , and δ_3 are shown in the top picture of the figure and the arcs δ_0 , δ_1 , and δ_2 are shown at the bottom). Again, each δ_i represents a vertex v_i of $\mathcal{C}(S_3)$ and again $d(v_0, v_3) = 3$ since δ_0 and δ_3 fill X_8 .

To see that the oriented geodesic v_0, v_1, v_2, v_3 is initially efficient we notice that δ_1 lies in a single region of X_8 determined by δ_0 and δ_3 and in that region it connects two marked points, one of which lies on δ_3 . It follows that the preimage of δ_1 in S_3 is a single nonseparating simple closed curve and if we cut S_3 along the preimages of δ_0 and δ_3 then this nonseparating curve becomes a single diagonal in a single polygonal region of the cut-open surface. From this it follows that v_0, v_1, v_2, v_3 is initially efficient.

A similar argument shows that the oriented geodesic v_3, v_2, v_1, v_0 is initially efficient. Indeed, the intersection of the arc δ_2 with each region of X_8 determined by δ_0 and δ_3 is a single arc. It follows that the preimage of δ_2 in S_3 intersects each polygonal region of S_3 in one or two arcs (depending on whether the corresponding arc in X_8 terminates at a marked point not contained in $\delta_0 \cup \delta_3$). As such, any reference arc in S_3 for the preimages of δ_0 and δ_3 can intersect the preimage of δ_2 in at most two points. The efficiency of v_0, v_1, v_2, v_3 follows.

We will now show that v_0, v_1, v_2, v_3 is not tight, in other words that v_0, v_1, v_2, v_3 is not contained in any tight multi-geodesic. Suppose $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ were a tight multi-geodesic containing v_0, v_1, v_2, v_3 . First of all, by definition we would have $\sigma_0 = v_0$ and $\sigma_3 = v_3$. Second, since (representatives of) v_0 and v_2 fill the complement of (a representative of) v_1 we must have that $\sigma_1 = v_1$. Now we notice that v_2 does not lie in a regular neighborhood of the union of representatives of $\sigma_1 = v_1$ and $\sigma_3 = v_3$ (since we can find an arc in X_8 that intersects δ_2 without intersecting δ_1 or δ_3). Therefore, for any choice of simplex σ_2 containing v_2 we will still have the property that σ_2 does not lie in a regular neighborhood of the union of representatives of σ_1 and σ_3 ; in particular, for any choice of σ_2 containing v_2 , the sequence $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ is not tight at σ_2 . Hence v_0, v_1, v_2, v_3 is not tight, as desired. Again the generalization to higher genus is clear. \square

3. EXISTENCE OF EFFICIENT PATHS

In this section we prove the main result of this paper, Theorem 1.1. The main point is to prove the existence of initially efficient geodesics (Proposition 3.2), and this will occupy most of the section. At the end we give the additional inductive argument for the existence of efficient geodesics (Theorem 1.1). Let $g \geq 2$ be fixed throughout.

3.1. Setup: a reducibility criterion. Our first goal is to recast the problem of finding initially efficient paths in terms of sequences of numbers; see Proposition 3.1 below.

Standard representatives and intersection sequences. Let v and w be vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. Let $v = v_0, \dots, v_n = w$ be an arbitrary path from v to w . We can choose representatives α_i of the v_i with the following properties:

- (1) each α_i is in minimal position with both α_0 and α_n ,
- (2) each intersection $\alpha_i \cap \alpha_{i+1}$ is empty, and
- (3) all triple intersections of the form $\alpha_i \cap \alpha_j \cap \alpha_k$ are empty.

To do this, we take the α_i to be geodesics with respect to some hyperbolic metric on S_g and then perform small isotopies to remove triple intersections. We say that such a collection of representatives for the v_i is *standard*. Note that we do not insist that α_i and α_j are in minimal position when $0 < i, j < n$ and $|i - j| > 1$.

Let γ be a reference arc for the standard set of representatives $\alpha_0, \dots, \alpha_n$, by which we mean that:

- (1) γ has its interior disjoint from $\alpha_0 \cup \alpha_n$,
- (2) γ has endpoints disjoint from $\alpha_1, \dots, \alpha_{n-1}$,

- (3) all triple intersections $\alpha_i \cap \alpha_j \cap \gamma$ are trivial for $i \neq j$, and
- (4) γ is in minimal position with each of $\alpha_1, \dots, \alpha_{n-1}$.

A reference arc for $\alpha_0, \dots, \alpha_n$ is automatically a reference arc for the triple $\alpha_0, \alpha_1, \alpha_n$ as in the introduction, but not the other way around. We will need to deal with this discrepancy in the proof of Proposition 3.2 below.

Denote the cardinality of $\gamma \cap (\alpha_1 \cup \dots \cup \alpha_{n-1})$ by N . Traversing γ in the direction of some chosen orientation, we record the sequence of natural numbers $\sigma = (j_1, j_2, \dots, j_N) \in \{1, \dots, n-1\}^N$ so that the i th intersection point of γ with $\alpha_1 \cup \dots \cup \alpha_{n-1}$ lies in α_{j_i} . We refer to σ as the *intersection sequence* of the α_i along γ .

Complexity of paths and reducible sequences. We define the *complexity* of an oriented path v_0, \dots, v_n in $\mathcal{C}(S)$ to be

$$\sum_{k=1}^{n-1} (i(v_0, v_k) + i(v_k, v_n)).$$

We say that a sequence σ of natural numbers is *reducible* under the following circumstances: whenever σ arises as an intersection sequence for a (standard set of representatives for) path v_0, \dots, v_n in $\mathcal{C}(S_g)$ there is another path v'_0, \dots, v'_n with $v'_0 = v_0$ and $v'_n = v_n$ and with smaller complexity. With this terminology in hand, the existence of initially efficient paths is a consequence of the following proposition.

Proposition 3.1. *Suppose σ is a sequence of elements of $\{1, \dots, n-1\}$. If σ has more than $n-1$ entries equal to 1, then σ is reducible.*

We can deduce the existence of initially efficient geodesics easily from Proposition 3.1.

Proposition 3.2. *Let $g \geq 2$. If v and w are vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$, then there exists an initially efficient geodesic from v to w .*

Proof of Proposition 3.2 assuming Proposition 3.1. Let v and w be vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. Since the complexity of any path from v to w is a natural number, there is a geodesic of minimal complexity. We will show that any geodesic from v to w that has minimal complexity must be initially efficient.

To this end, we consider an arbitrary geodesic $v = v_0, \dots, v_n = w$ and we assume that it is not initially efficient. In other words there is a set of representatives $\alpha_0, \alpha_1, \alpha_n$ for v_0, v_1, v_n that are in minimal position and a reference arc γ for $\alpha_0, \alpha_1, \alpha_n$ with $|\alpha_1 \cap \gamma| > n-1$.

We can extend the triple $\alpha_0, \alpha_1, \alpha_n$ to a set of standard representatives $\alpha_0, \dots, \alpha_n$ for the whole geodesic v_0, \dots, v_n . What is more, we may assume that γ is a reference arc for this full set of representatives $\alpha_0, \dots, \alpha_n$.

Indeed, if γ is not in minimal position with some α_i with $2 \leq i \leq n-1$ then by an adaptation of the usual bigon criterion for simple closed curves we have that γ and α_i cobound an embedded bigon; if we choose an innermost such bigon (with respect to γ) and push the corresponding α_i across, then we can eliminate the bigon without creating any new points of intersection between γ with any α_j or between

any two α_j . (Alternatively, as in the introduction, we can assume that each α_i with $1 \leq i \leq n-1$ is a straight line segment in each polygon determined by α_0 and α_n and we can take γ to be any straight line segment; this procedure always yields a γ that is in minimal position with each α_i).

Since we did not change α_1 , the new intersection sequence of $\alpha_0, \dots, \alpha_n$ with γ still has more than $n-1$ entries equal to 1. By Proposition 3.1, the sequence σ is reducible. This implies that v_0, \dots, v_n does not have minimal complexity, and we are done. \square

Notice that the approach established in Proposition 3.1 disregards all information about a path in $\mathcal{C}(S_g)$ except its intersection sequences. For instance, we will not need to concern ourselves with how the strands of the α_i are connected outside of a neighborhood of γ .

We will prove Proposition 3.1 in three stages. First, in Section 3.2 we describe a normal form for sequences of natural numbers (Lemma 3.3 below) and also describe an associated diagram for the normal form called the dot graph. Next in Section 3.3 we will show that if the dot graph exhibits certain geometric features—empty boxes and hexagons—then the sequence is reducible (Lemma 3.4). Finally in Section 3.4 we will show that any sequence in normal form that does not satisfy Proposition 3.1 has a dot graph exhibiting either an empty box or an empty hexagon, hence proving Proposition 3.1.

3.2. Stage 1: Sawtooth form and the dot graph. The main goal of this section is to give a normal form for sequences of natural numbers that interacts well with our notion of reducibility. We also describe a way to diagram sequences in normal form called the dot graph.

Sawtooth form. We say that a sequence (j_1, j_2, \dots, j_k) of natural numbers is in *sawtooth form* if

$$j_i < j_{i+1} \implies j_{i+1} = j_i + 1.$$

An example of a sequence in sawtooth form is $(1, 2, 2, 3, 4, 3, 4, 3, 4, 2, 3, 4, 5)$. If a sequence of natural numbers is in sawtooth form, we may consider its *ascending sequences*, which are the maximal subsequences of the form $k, k+1, \dots, k+m$. In the previous example, the ascending sequences are $(1, 2)$, $(2, 3, 4)$, $(3, 4)$, $(3, 4)$, and $(2, 3, 4, 5)$.

Lemma 3.3. *Let σ be an intersection sequence. There exists an intersection sequence τ in sawtooth form so that τ differs from σ by a permutation of its entries and so that σ is reducible if and only if τ is.*

Proof. Suppose $\sigma = (j_1, \dots, j_N)$ is the intersection sequence for a set of standard representatives $\alpha_0, \dots, \alpha_n$ along an arc $\gamma \subseteq \alpha_n \setminus \alpha_0$. The basic idea we will use is that if $|j_i - j_{i+1}| > 1$, then we can modify α_{j_i} and $\alpha_{j_{i+1}}$ to new curves α'_{j_i} and $\alpha'_{j_{i+1}}$ so that the new curves still form a set of standard representatives for the same path and so that the new intersection sequence along γ differs from σ by a transposition of the consecutive terms j_i and j_{i+1} ; see Figure 7. We call this the resulting modification of σ a *commutation*.

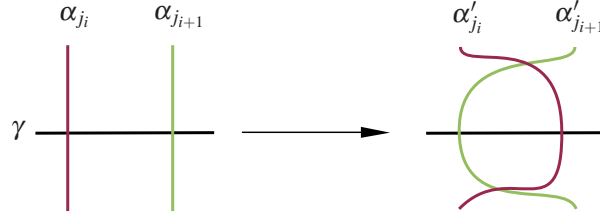


FIGURE 7. A commutation

It suffices to show that if a sequence σ is not in sawtooth form, then it is possible to perform a finite sequence of commutations so that the resulting sequence τ is in sawtooth form. Indeed, the sequence τ appears as an intersection sequence for a particular path in $\mathcal{C}(S_g)$ if and only if σ does (the key point is that commutations never result in a nonempty intersection of the form $\alpha_i \cap \alpha_{i+1}$).

We say that σ fails to be in sawtooth form at the index i if $j_{i+1} > j_i + 1$. Let $k = k(\sigma)$ be the highest index at which σ fails to be in sawtooth form, and say that k is zero if σ is in sawtooth form. Assuming $k > 0$, we will show that we can modify σ by a sequence of commutations so that the highest index where the resulting sequence fails to be in sawtooth form is strictly less than k .

We decompose σ into a sequence of subsequences of σ , namely,

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

where σ_2 is the singleton (j_k) and σ_3 is the longest subsequence of σ starting from the $(k+1)$ st term so that each term is greater than $j_k + 1$. The sequences σ_1 and σ_4 are thus determined, and one or both might be empty.

By a series of commutations, we can modify σ to the sequence

$$\sigma' = (\sigma_1, \sigma_3, \sigma_2, \sigma_4).$$

We claim that $k(\sigma') < k(\sigma)$. Since the length of σ_1 is $k-1$, it is enough to show that the subsequence $(\sigma_3, \sigma_2, \sigma_4)$ is in sawtooth form.

By the definition of k , we know that σ_3 is in sawtooth form. Next, the last term of σ_3 is greater than $j_k + 1$ and the first (and only) term of σ_2 is j_k , and so these terms satisfy the definition of sawtooth form. We know $\sigma_2 = (j_k)$ and the first term of σ_4 , call it j , is at most $j_k + 1$, and so these terms are also in sawtooth form. Finally, the subsequence σ_4 is in sawtooth form by the definition of k . This completes the proof. \square

Dot graphs. It will be useful to draw the graph in $\mathbb{R}_{\geq 0}^2$ of a given sequence of natural numbers, where the sequence is regarded as a function $\{1, \dots, N\} \rightarrow \mathbb{N}$. The points of the graph of a sequence σ will be called *dots*. We decorate the graph by connecting the dots that lie on a given line of slope 1; these line segments will be called *ascending segments*. The resulting decorated graph will be called the *dot graph* of σ and will be denoted $G(\sigma)$; see Figure 8.

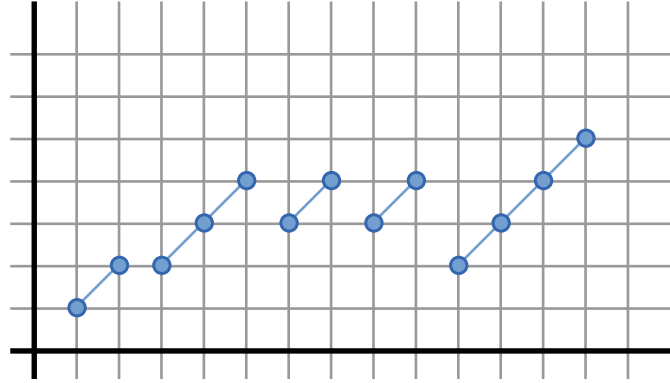


FIGURE 8. Example of dot graph of a sequence in sawtooth form

3.3. Stage 2: Dot graph polygons and surgery. The goal of this section is to describe certain geometric shapes that can arise in a dot graph, and then to prove that if the dot graph $G(\sigma)$ admits one of these shapes then the sequence σ is reducible (Lemma 3.4).

Dot graph polygons. We say that a polygon in the plane is a *dot graph polygon* if

- (1) the edges all have slope 0 or 1,
- (2) the edges of slope 0 have nonzero length, and
- (3) the vertices all have integer coordinates.

The edges of slope 1 in a dot graph polygon are called *ascending edges* and the edges of slope 0 are called *horizontal edges*.

Let σ be a sequence of natural numbers in sawtooth form. A dot graph polygon is a σ -*polygon* if:

- (1) the vertices are dots of $G(\sigma)$ and
- (2) the ascending edges are contained in ascending segments of $G(\sigma)$.

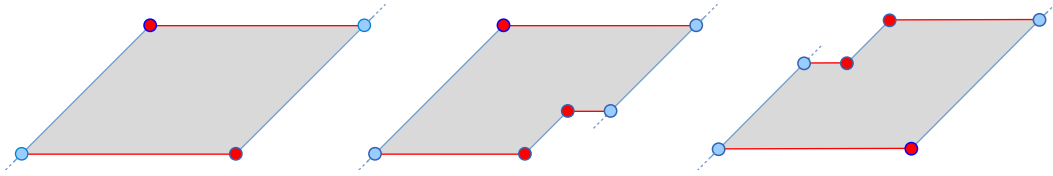


FIGURE 9. A box, a hexagon of type 1, and a hexagon of type 2; the red (darker) dots are required to be endpoints of ascending segments, while the blue (lighter) dots may or may not be endpoints

A *box* in $G(\sigma)$ is a σ -quadrilateral P with the following two properties:

- (1) the leftmost ascending edge contains the highest point of some ascending segment of $G(\sigma)$ and
- (2) the rightmost ascending edge contains the lowest point of some ascending segment of $G(\sigma)$.

We will also need to deal with hexagons. Up to translation and changing the edge lengths, there are four types of dot graph hexagons; two have an acute exterior angle, and we will not need to consider these. Notice that a dot graph hexagon necessarily has a leftmost ascending edge, a rightmost ascending edge, and a middle ascending edge. This holds even for degenerate hexagons since horizontal edges are required to have nonzero length.

A *hexagon of type 1* in $G(\sigma)$ is a σ -hexagon where:

- (1) no exterior angle is acute,
- (2) the middle ascending edge is an entire ascending segment of $G(\sigma)$, and
- (3) the minimum of the middle ascending edge equals the minimum of the leftmost ascending edge,
- (4) the leftmost ascending edge contains the highest point of an ascending segment of $G(\sigma)$.

Similarly, a *hexagon of type 2* in $G(\sigma)$ is a σ -hexagon that satisfies the first two conditions above and the following third and fourth conditions:

- (3') the maximum of the middle ascending edge equals the maximum of the rightmost ascending edge,
- (4') the rightmost ascending edge contains the lowest point of an ascending segment of $G(\sigma)$.

See Figure 9 for pictures of boxes and hexagons of types 1 and 2.

The following lemma is the main goal of this section. We say that a horizontal edge of a σ -polygon is *pierced* if its interior intersects $G(\sigma)$. Also, we say that a σ -polygon is *empty* if it there are no points of $G(\sigma)$ in its interior.

Lemma 3.4. *Suppose that σ is a sequence of natural numbers in sawtooth form and that $G(\sigma)$ has an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2. Then σ is reducible.*

Before we prove Lemma 3.4, we need to introduce another topological tool, surgery on curves.

Surgery. Let α be a simple closed curve in a surface and let γ be an oriented arc so that α and γ are in minimal position. We can form a new curve α' from α by performing surgery along γ as follows. We first remove from α small open neighborhoods of two points of $\alpha \cap \gamma$ that are consecutive along γ . What remains of α is a pair of arcs; we can connect the endpoints of either arc by another arc δ that lies in a small neighborhood of γ in order to create the new simple closed curve α' (the other arc of α is discarded); see Figure 10.

We draw a neighborhood of γ in the plane so that γ is a horizontal arc oriented to the right. We say that α' is obtained from α by $++$, $+-$, $-+$, or $--$ surgery along γ ; the first symbol is $+$ or $-$ depending on whether the first endpoint of δ (as measured by the orientation of γ) lies above γ or below, and similarly for the second symbol.

In general, for a given pair of intersection points of a curve α with γ , exactly two of the four possible surgeries result in a simple closed curve. If we orient α ,

then the two intersection points of α with γ can either agree or disagree. If they agree, then the $+-$ and $-+$ surgeries, the *odd surgeries*, result in a simple closed curve, and if they disagree, the $++$ and $--$ surgeries, the *even surgeries*, result in a simple closed curve.

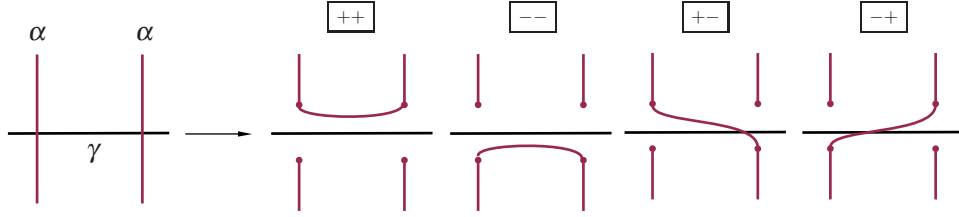


FIGURE 10. The four types of surgery on a curve along an arc

These surgeries will of course only be of use to us if the curve α' is an essential simple closed curve in S . One variant of the well-known bigon criterion is that a curve α and an arc γ are in minimal position if and only if every closed curve formed from α and γ as above is essential. Indeed, the proof in the case where α and γ are both curves (see [1, Proposition 3.10]) can be adapted to this case. Thus our α' is essential.

Surfaces with boundary. In order to show that our surgered curves are essential, we used a version of the bigon criterion. This bigon criterion is exactly what fails in the case of surfaces with boundary. For instance, suppose that the surface S has at least two boundary components and consider a simple closed curve α that cuts off a pair of pants in S . If γ is an arc that intersects α in two points then both of the curves obtained by surgering α along γ are homotopic to components of the boundary of S , neither of which represents a vertex of $\mathcal{C}(S)$.

We now use the surgeries described above to prove that a dot graph with an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2 corresponds to a sequence that is reducible.

Proof of Lemma 3.4. Suppose that σ appears as an intersection sequence for a reference arc γ for a set of standard representatives $\alpha_0, \dots, \alpha_n$ for a path v_0, \dots, v_n in $\mathcal{C}(S_g)$. We need to replace the α_i with new curves α'_i so that the resulting path from v_0 to v_n has smaller complexity. We treat the three cases in turn, according to whether $G(\sigma)$ has an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2.

Suppose $G(\sigma)$ has an empty, unpierced box P . By the definitions of sawtooth form and empty boxes there are no ascending edges of $G(\sigma)$ in the vertical strip between the two ascending edges of P , that is, the dots of P correspond to a consecutive sequence of intersections along γ :

$$\alpha_k, \dots, \alpha_{k+m}, \quad \alpha_k, \dots, \alpha_{k+m}$$

where $1 \leq k \leq k+m \leq n-1$.

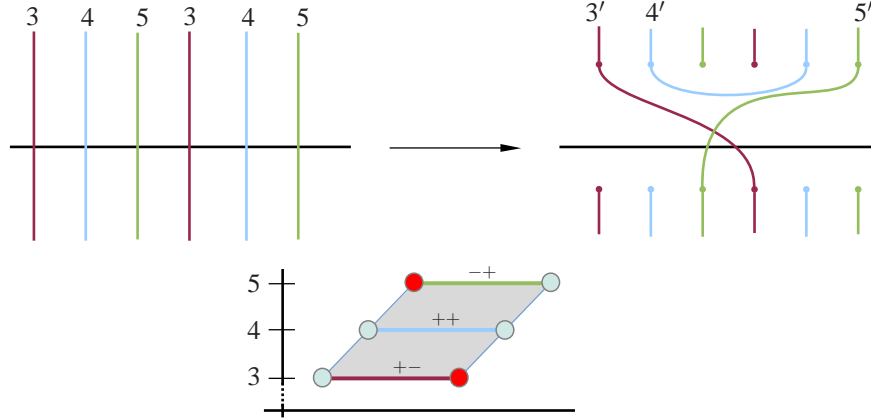


FIGURE 11. An example of a set of surgeries as in the box case of Lemma 3.4

First, for $i \notin \{k, \dots, k+m\}$ we set $\alpha'_i = \alpha_i$. We then define $\alpha'_k, \dots, \alpha'_{k+m}$ inductively: for $i = k, \dots, k+m$, the curve α'_i is obtained by performing surgery along γ between the two points of $\alpha_i \cap \gamma$ corresponding to dots of P and the surgeries are chosen so that they form a path in the directed graph in Figure 12 (of course for each i we must choose one of the two surgeries that results in a closed curve).

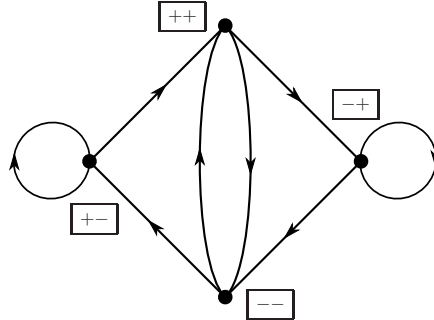


FIGURE 12. The directed graph used in the proof of Lemma 3.4

The vertices of the graph in Figure 12 correspond to the four types of surgeries: $++$, $+-$, $-+$, and $--$, and the rule is that the second sign of the origin of a directed edge is the opposite of the first sign of the terminus. Since every vertex has one outgoing arrow pointing to an even surgery and one outgoing arrow pointing to an odd surgery, the desired sequence of surgeries exists; in fact it is completely determined by the choice of surgery on α_k , and so there are exactly two possible sequences. See Figure 11 for an example of this procedure; there we perform $+-$ surgery on α_3 , then $++$ surgery on α_4 , then $-+$ surgery on α_5 .

For $0 \leq i \leq n$, let v'_i be the vertex of $\mathcal{C}(S_g)$ represented by α'_i . We need to check that the v'_i certify the reducibility of σ , namely that

- (1) $v'_0 = v_0$ and $v'_n = v_n$,
- (2) each v'_i is connected to v'_{i+1} by an edge in $\mathcal{C}(S_g)$, and

(3) the complexity of v'_0, \dots, v'_n is strictly smaller than that of v_0, \dots, v_n .

The first condition holds because $1 \leq k \leq k+m \leq n-1$. The second condition holds because each intersection $\alpha_i \cap \alpha_{i+1}$ is empty and the surgeries do not create new intersections. For the third condition, we claim that something stronger is true, namely, that

$$i(v_0, v'_i) + i(v'_i, v_n) \leq i(v_0, v_i) + i(v_i, v_n)$$

for all i and that

$$i(v_0, v'_k) + i(v'_k, v_n) < i(v_0, v_k) + i(v_k, v_n).$$

Indeed, if we consider the polygonal decomposition of S_g determined by $\alpha_0 \cup \alpha_n$ we see that when we surger two strands of some α_i along γ we create no new intersections with $\alpha_0 \cup \alpha_n$ and we remove two intersections with $\alpha_0 \cup \alpha_n$ (we might also create a bigon, but this would only help our case). Since we performed at least one surgery—on α_k —our claim is proven.

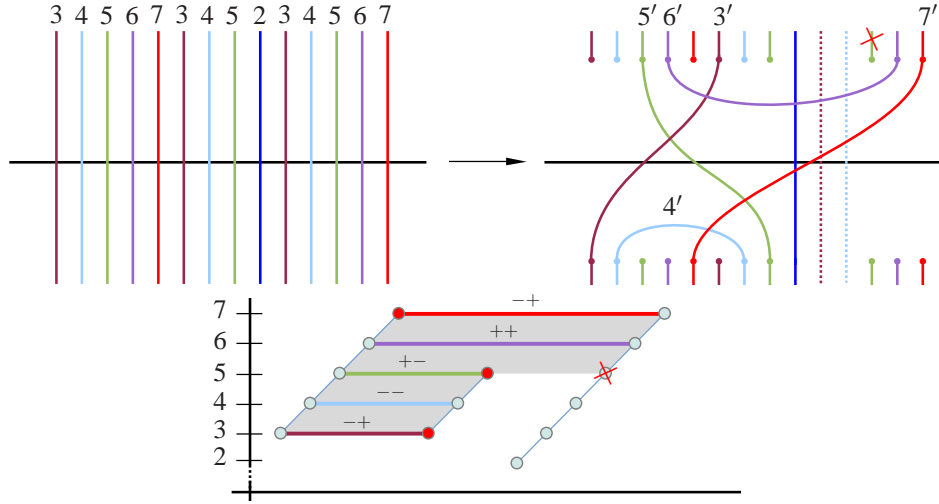


FIGURE 13. An example of a set of surgeries as in the hexagon case of Lemma 3.4

The cases of empty, unpierced hexagons of types 1 and 2 are similar, but one new idea is needed. These two cases are almost identical, and so we will only treat the first case, that is, we suppose $G(\sigma)$ has an empty, unpierced hexagon P of type 1. By the definition of sawtooth form and the definition of an empty, unpierced hexagon of type 1, there are no ascending segments of $G(\sigma)$ in the vertical strip between the leftmost and middle ascending edges of P and any ascending segments of $G(\sigma)$ that lie in the vertical strip between the middle and rightmost ascending segments have their highest point strictly below the lower-right horizontal edge of P . It follows that the dots of P correspond to a sequence of intersections along γ of the following form

$$\alpha_k, \dots, \alpha_{k+m}, \quad \alpha_k, \dots, \alpha_{k+l}, \quad \alpha_{j_1}, \dots, \alpha_{j_p}, \quad \alpha_{k+l}, \dots, \alpha_{k+m}$$

where $1 \leq k \leq k + \ell \leq k + m \leq n - 1$, $p \geq 0$, and each $j_i < \alpha_{k+\ell}$. See Figure 13 for an example where $k = 3$, $\ell = 2$, $m = 4$, and $p = 0$.

Again, for each α_i with $i \notin \{k, \dots, k + m\}$ we set $\alpha'_i = \alpha_i$. Each of the remaining α_i corresponds to exactly two dots in P except for $\alpha_{k+\ell}$, which corresponds to three. Let $\alpha'_{k+\ell}$ be the curve obtained from $\alpha'_{k+\ell}$ via surgery along γ between the first two (leftmost) points of $\alpha'_{k+\ell} \cap \gamma$ corresponding to dots of P and satisfying the following property: $\alpha'_{k+\ell}$ does not contain the arc of $\alpha_{k+\ell}$ containing the third (rightmost) point of $\alpha_{k+\ell} \cap \gamma$ corresponding to a dot of P . As always, there are two choices of surgery given two consecutive points of $\alpha_{k+\ell} \cap \gamma$; one contains this third intersection point and one does not.

We then define $\alpha'_{k+\ell-1}, \dots, \alpha'_k$ inductively as before using the diagram above (notice the reversed order), and finally we define $\alpha'_{k+\ell+1}, \dots, \alpha'_{k+m}$ inductively as before.

By our choice of $\alpha'_{k+\ell}$, we have that $\alpha'_{k+\ell} \cap \alpha'_{k+\ell+1} = \emptyset$, as required; indeed, we eliminated the strand of $\alpha'_{k+\ell}$ that was in the way between the two strands of $\alpha_{k+\ell+1}$ being surgered. Also, since each j_i is strictly less than $k + \ell$, the curves $\alpha'_{k+\ell+1}, \dots, \alpha'_{k+m}$ satisfy the condition that $\alpha'_i \cap \alpha'_{i+1} = \emptyset$. The other conditions in the definition of a reducible sequence are easily verified as before. This completes the proof of the lemma. \square

3.4. Stage 3: Innermost polygons. In this section we will put together Lemmas 3.3 and 3.4 in order to prove Proposition 3.1. We begin with two lemmas.

Lemma 3.5. *If a dot graph $G(\sigma)$ contains a box P pierced in exactly one edge, then it contains an unpierced box.*

Proof. Denote the ascending edges of P by e and f . There is an ascending segment e' intersecting the interior of exactly one of the two horizontal edges of P ; we choose e' to be rightmost if it intersects the bottom edge of P and leftmost if it intersects the top edge. Either way, we find a box P' pierced in at most one edge and where one ascending edge is contained in e' and the other ascending edge is contained in P . The box P' has horizontal edges strictly shorter than those of P . Therefore, we may repeat the process until it eventually terminates, at which point we find the desired unpierced box. \square

Lemma 3.6. *Among all unpierced boxes and hexagons of type 1 and 2 in a dot graph $G(\sigma)$, an innermost unpierced box or hexagon of type 1 or 2 is empty.*

Proof. We treat the three cases separately. First suppose that P is an unpierced box that is not empty. We will show that P either contains another unpierced box or an unpierced hexagon of type 1. Let e be an ascending segment contained in the interior of P . We choose e so that $\max(e)$ is maximal among all such ascending segments, and we further choose e to be rightmost among all ascending segments with maximum equal to $\max(e)$.

There is a unique (possibly degenerate) hexagon P' of type 1 with one edge equal to e , and the other two edges contained in the ascending edges of P ; see the left-hand side of Figure 14. If P' is unpierced, we are done, so assume that P' is pierced.

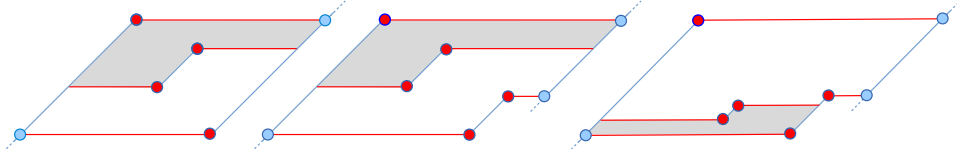


FIGURE 14. Inside a box, inside a hexagon, inside a hexagon

By construction, the top horizontal edge of P' and the lower-right horizontal edge of P' are unpierced. Suppose that the interior of the lower-left horizontal edge of P' were pierced. Let e' be the rightmost ascending segment of $G(\sigma)$ that pierces this edge of P' . By the choice of e , we have that $\max(e') \leq \max(e)$, and so there is a box pierced in at most one edge whose ascending edges are contained in e' and e . By Lemma 3.5, there is an unpierced box contained in this pierced box, and so P is not innermost.

The second case is where P is an unpierced hexagon of type 1. Again suppose that P is not empty. Let e be an ascending segment contained in the interior of P that has the largest maximum $\max(e)$ over all such segments and is rightmost among all such ascending segments. Let m denote the middle ascending edge of P . It follows from the fact that σ is in sawtooth form that there are no ascending segments of $G(\sigma)$ that lie inside P and to the right of m ; so e lies to the left of m . We now treat two subcases, depending on whether $\max(e) > \max(m)$ or not.

If $\max(e) > \max(m)$, there is a maximal hexagon P' of type 1 with ascending edges contained in $P \cup e$ as in the middle picture of Figure 14. By the same argument as in the previous case, P' is either unpierced or it contains an unpierced box.

If $\max(e) \leq \max(m)$, the argument is similar. There is a hexagon P' of type 2 as shown in the right-hand side of Figure 14. The topmost edge of P' is unpierced by the choice of e . The bottom edge of P' is unpierced since it is a horizontal edge for P , which is unpierced. And if the third horizontal edge of P' were pierced, we could find a box pierced in at most one edge, hence an unpierced box, as in the previous cases. It follows that P' is unpierced and again P is not innermost.

The third and final case is where P is an unpierced hexagon of type 2. This is completely analogous to the previous case; in fact, if we rotate the two pictures from the type 1 case by π we obtain the required pictures for the type 2 case. \square

We can now use the two previous lemmas to prove Proposition 3.1.

Proof of Proposition 3.1. Let σ be a sequence of elements of $\{1, \dots, n-1\}$. By Lemma 3.3 we may assume that σ is in sawtooth form without changing the number of entries equal to 1; call this number k . Let e_1, \dots, e_k denote the ascending segments of $G(\sigma)$ with minimum equal to 1, ordered from left to right.

If $\max(e_{i+1}) < \max(e_i)$ for all i , then since $\max(e_1) \leq n-1$ it follows that $k \leq n-1$. Therefore, it suffices to show that if $\max(e_{i+1}) \geq \max(e_i)$ for some i then σ is reducible.

Suppose then that $\max(e_{i+1}) \geq \max(e_i)$ for some i . The first step is to show that $G(\sigma)$ has an unpierced box. Let e be the first ascending segment (from left to right) that appears after e_i and has $\max(e) \geq \max(e_i)$. Because $\min(e) \geq \min(e_i) = 1$, there is evidently a (possibly degenerate) box P with two edges contained in e_i and e and two horizontal edges with heights $\min(e)$ and $\max(e_i)$. By the definition of e , the interior of the upper horizontal edge of P is disjoint from $G(\sigma)$, so P is pierced in at most one edge. By Lemma 3.5, P contains an unpierced box.

Let P now be an innermost unpierced box or hexagon of type 1 or 2; such P exists because each σ -polygon contains a finite number of dots of $G(\sigma)$ and a polygon contained inside another polygon contains a fewer number of dots. By Lemma 3.6, the polygon P is empty. By Lemma 3.4, σ is reducible. \square

3.5. From initially efficient geodesics to efficient geodesics. At this point we have established the existence of initially efficient geodesics (Proposition 3.2). It remains to establish the existence of efficient geodesics (Theorem 1.1).

Total complexity. For an oriented path q in $\mathcal{C}(S_g)$ with vertices w_0, \dots, w_n define the complexity $\kappa(q)$ as before:

$$\kappa(q) = \sum_{k=1}^{n-1} (i(w_0, w_k) + i(w_k, w_n)).$$

Next, for an oriented path p with vertices v_0, \dots, v_n , let p_1 be the oriented path v_n, \dots, v_{n-3} and let p_k be the oriented path v_{n-k-1}, \dots, v_n for $2 \leq k \leq n-1$. We will relabel the vertices of p_k as w_0, \dots, w_{n_k} . The *total complexity* of a path p is the ordered $(n-1)$ -tuple:

$$\hat{\kappa}(p) = (\kappa(p_1), \dots, \kappa(p_{n-1})).$$

We order the set \mathbb{N}^{n-1} —hence the set of total complexities—lexicographically.

Proof of Theorem 1.1. Let v and w be vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. We claim that any geodesic from v to w that has minimal total complexity must be efficient.

Let p be an arbitrary geodesic $v = v_0, \dots, v_n = w$ and assume that p is not efficient. In other words, one of the corresponding paths p_k with vertices w_0, \dots, w_{n_k} is not initially efficient. This is the same as saying that there is a set of representatives $\beta_0, \beta_1, \beta_{n_k}$ for w_0, w_1, w_{n_k} that are in minimal position and a reference arc γ with $|\beta_1 \cap \gamma| > n_k - 1$.

As in the proof of Proposition 3.2 we can extend the triple $\beta_0, \beta_1, \beta_{n_k}$ to a full standard set of representatives $\beta_0, \dots, \beta_{n_k}$ for p_k . And as in that proof there are surgeries that reduce the complexity of p_k . The curves obtained by these surgeries not only give a new path between the endpoints of p_k , but they also give rise to a new path between v and w .

The key observation here is that, by our choice of the order of the p_i , the surgeries used in modifying p_k do not increase the complexity of any p_i with $i < k$. Indeed, these surgeries do not increase the intersection between any of the curves $\beta_0, \dots, \beta_{n_k}$ and all of the vertices of p used in the computation of $\kappa(p_i)$ with $i < k$ are already vertices of p_k , namely, the vertices represented by $\beta_0, \dots, \beta_{n_k}$. The theorem follows. \square

3.6. An improved algorithm in a special case. We end this section by stating and proving the alternate version of the efficient geodesic algorithm that was used in the example at the start of Section 2.1. This proposition is equivalent to the main theorem (Theorem 1.1) of the first version of this paper [4].

Proposition 3.7. *Suppose v and w are vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. Let α and β be representatives of α and β that are in minimal position. Then there is a geodesic $v = v_0, \dots, v_n = w$ and a representative α_1 of v_1 so that the number of intersections of α_1 with each arc of $\beta \setminus \alpha$ is at most $d(v, w) - 2$.*

Proof. The proof is essentially the same as the proof of Theorem 1.1. The only added observation is that, since γ is a subset of β , every intersection sequence can be taken to have entries in $\{1, \dots, n-2\}$ instead of $\{1, \dots, n-1\}$. \square

Note that in the special case that vertices v and w have representatives α and β that cut the surface into rectangles and hexagons only (e.g. the example of Section 2), then every reference arc is parallel to a reference arc as in Proposition 3.7, and so in this case there are geodesics that are extra efficient in the sense that the intersection of a representative of v_1 with any reference arc is at most $n-2$ instead of $n-1$.

APPENDIX A. WEBB'S ALGORITHM

In this appendix we give an exposition of Webb's algorithm for computing distance in $\mathcal{C}(S)$. As with the efficient geodesic algorithm we will make the inductive hypothesis that for some $n \geq 2$ we have an algorithm to determine if the distance between two vertices is $0, \dots, n-1$ and we would like to give an algorithm for determining if the distance between two vertices is n . First we introduce an auxiliary tool, the arc complex for a surface with boundary.

Arc complex. Let F be a compact surface with nonempty boundary. The arc complex $\mathcal{A}(F)$ is the simplicial complex with k -simplices corresponding to $(k+1)$ -tuples of homotopy classes of essential arcs in F with pairwise disjoint representatives. Here, homotopies are allowed to move the endpoints of an arc along ∂F , and an arc is essential if it is not homotopic into ∂F .

The algorithm. A maximal simplex of $\mathcal{A}(F)$ can be regarded as a triangulation of the surface obtained from F by collapsing each component of the boundary to a point. If F is a compact, orientable surface of genus g with m boundary components, then the number of edges in any such triangulation is $6g + 3m - 6$.

Let v and w be two vertices of $\mathcal{C}(S)$ with $d(v, w) \geq 3$. As in the efficient geodesic algorithm, it suffices by the induction hypothesis to list all candidates for vertices v_1 on a tight geodesic $v = v_0, \dots, v_n = w$. Since there are finitely many vertices in each simplex of $\mathcal{C}(S)$ it further suffices to list all candidates for simplices σ_1 on a tight multigeodesic $v = \sigma_0, \dots, \sigma_n = w$.

Suppose we have such a tight multigeodesic $v = \sigma_0, \dots, \sigma_n = w$. We can choose representatives α_i of the σ_i so that $\alpha_i \cap \alpha_{i+1} = \emptyset$ for all i and so that each α_i lies in minimal position with α_0 . If we cut S along α_0 , we obtain a compact surface S' , some of whose boundary components correspond to α_0 .

For each $i > 1$, the representative α_i gives a collection of disjoint arcs in S' and hence a simplex τ_i of $\mathcal{A}(S')$ (some arcs of α_i might be parallel and these get identified in $\mathcal{A}(S')$). For $i \geq 3$, the collection of arcs is filling, which means that when we cut S' along these arcs we obtain a collection of disks and boundary-parallel annuli, and we say that the corresponding simplex of $\mathcal{A}(S')$ is filling.

Since there is a unique configuration for α_n and α_0 in minimal position, there is a unique possibility for τ_n . As $\tau_n \cup \tau_{n-1}$ is contained in a simplex of the arc complex of S' and since τ_n is filling, there are finitely many possibilities for τ_{n-1} (and we can explicitly list them). This is the key point: there are infinitely many vertices of $\mathcal{C}(S)$ that correspond to any given simplex in the arc complex, but there are finitely many choices for the simplex itself.

Because τ_i is filling whenever $i \geq 3$, we can continue this process inductively, and explicitly list all possibilities for τ_2 . Now, by the definition of a tight multi-geodesic in $\mathcal{C}(S)$, the simplex σ_1 is represented by the union of the essential components of the boundary of a regular neighborhood of $\alpha_0 \cup \alpha_2$. Equivalently, any such σ_1 is given by a regular neighborhood of the union of $\partial S'$ with a representative of τ_2 . Hence there are finitely many (explicitly listable) possibilities for σ_1 , as desired.

A bound on the number of candidates. In the introduction we stated that the number of candidate simplices σ_1 produced by Webb's algorithm when $d(v, w) = n$ is bounded above by

$$2^{(72g+12)\min\{n-2, 21\}}(2^{6g-6} - 1).$$

We will now explain this bound; we are grateful to Richard Webb for supplying us with the details.

We can think of the sequence τ_n, \dots, τ_3 as a path in the filling multi-arc complex, that is, the simplicial complex whose vertices are simplices of $\mathcal{A}(S')$ whose geometric realizations fill S' and whose edges correspond to simplices with geometric intersection number zero. Then we obtain τ_2 by extending this path by one more edge and taking some nonempty subset of the simplex of $\mathcal{A}(S')$ represented by the endpoint $\hat{\tau}_2$ of this extended path.

Webb proved that the degree of an arbitrary vertex of this filling multi-arc complex is bounded above by 2^{72g+12} (this is for the case where we start with a closed surface of genus g and cut along a single simple closed curve, as above); see his paper [19]. Our extended path from τ_n to $\hat{\tau}_{n-2}$ has length $n - 2$ and so this a priori gives a bound of $2^{(72g+12)(n-2)}$ for the number of possibilities for $\hat{\tau}_2$. However, there is a version of the bounded geodesic image theorem which tells us that, because the τ_i arise from a geodesic in $\mathcal{C}(S)$, the actual distance in the filling multi-arc complex between τ_n and $\hat{\tau}_2$ is bounded above by 21. This gives the first multiplicand in the desired bound. The second multiplicand comes from the number of ways of choosing a nonempty sub-simplex τ_2 of $\hat{\tau}_2$. The number of vertices of τ_2 is bounded above by $6g - 6$, and so there are $2^{6g-6} - 1$ ways to choose τ_2 from $\hat{\tau}_2$.

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